数列（5）　「部分分数」「S−rS」

部分分数に分解する。 $\dfrac{1}{\underline{n}\,(\underline{n+d})} = \dfrac{1}{d}\left(\dfrac{1}{n} - \dfrac{1}{n+d}\right)$

差が d

例題1　次の和 S を求めよ。

(1) $S=\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\cdots\cdots+\dfrac{1}{(2n-1)(2n+1)}$

$= \dfrac{1}{2}\left\{\left(1-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{5}\right)+\cdots+\left(\dfrac{1}{2n-1}-\dfrac{1}{2n+1}\right)\right\}$

$= \dfrac{1}{2}\left(1-\dfrac{1}{2n+1}\right) = \dfrac{1}{2}\times\dfrac{2n}{2n+1} = \boxed{\dfrac{n}{2n+1}}$

(2) $S=\dfrac{1}{\sqrt{1}+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\cdots\cdots+\dfrac{1}{\sqrt{n}+\sqrt{n+1}}$

$= \dfrac{\sqrt{1}-\sqrt{2}}{-1}+\dfrac{\sqrt{2}-\sqrt{3}}{-1}+\cdots+\dfrac{\sqrt{n}-\sqrt{n+1}}{-1}$

$= \dfrac{\sqrt{1}-\sqrt{n+1}}{-1} = \boxed{\sqrt{n+1}-1}$

1　次の和 S を求めよ。

(1) $S=\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\cdots\cdots+\dfrac{1}{(3n-2)(3n+1)}$

$= \dfrac{1}{3}\left\{\left(1-\dfrac{1}{4}\right)+\left(\dfrac{1}{4}-\dfrac{1}{7}\right)+\cdots+\left(\dfrac{1}{3n-2}-\dfrac{1}{3n+1}\right)\right\}$

$= \dfrac{1}{3}\left(1-\dfrac{1}{3n+1}\right)$

$= \dfrac{1}{3}\times\dfrac{3n}{3n+1} = \boxed{\dfrac{n}{3n+1}}$

(2) $S=\displaystyle\sum_{k=1}^{n}\dfrac{1}{\sqrt{k+2}+\sqrt{k+3}}$

$= \displaystyle\sum_{k=1}^{n}\dfrac{\sqrt{k+2}-\sqrt{k+3}}{-1}$

$= \dfrac{\sqrt{3}-\sqrt{4}}{-1}+\dfrac{\sqrt{4}-\sqrt{5}}{-1}+\cdots+\dfrac{\sqrt{n+2}-\sqrt{n+3}}{-1}$

$= \dfrac{\sqrt{3}-\sqrt{n+3}}{-1} = \boxed{\sqrt{n+3}-\sqrt{3}}$

例題2 次の和 S を求めよ。

$$S=1\cdot 1+3\cdot 3+5\cdot 3^2+\cdots\cdots+(2n-1)\cdot 3^{n-1}$$

$$-)\quad 3S = 1\cdot 3+3\cdot 3^2+\cdots+(2n-3)3^{n-1}+(2n-1)3^n$$

$$-2S = 1+2\cdot 3+2\cdot 3^2+\cdots+2\cdot 3^{n-1}-(2n-1)3^n$$

$$-2S = 1+2(3+3^2+\cdots+3^{n-1})-(2n-1)3^n$$

$$= 1+2\times\frac{3(3^{n-1}-1)}{3-1}+(-2n+1)3^n$$

$$= 1+3^n-3+(-2n+1)3^n$$

$$-2S = (-2n+2)3^n-2$$

$$S = \boxed{(n-1)3^n+1}$$

2 次の和 S を求めよ。

$$S=1\cdot 1+2\cdot 5+3\cdot 5^2+4\cdot 5^3+\cdots\cdots+n\cdot 5^{n-1}$$

$$-)\quad 5S = 1\cdot 5+2\cdot 5^2+3\cdot 5^3+\cdots+(n-1)5^{n-1}+n\cdot 5^n$$

$$-4S = 1+5+5^2+5^3+\cdots+5^{n-1}-n\cdot 5^n$$

$$= \frac{5^n-1}{5-1}-n\cdot 5^n$$

$$= \frac{5^n-1-4n\cdot 5^n}{4}$$

$$-4S = \frac{(-4n+1)5^n-1}{4}$$

$$S = \boxed{\frac{(4n-1)5^n+1}{16}}$$